

Orbital Functional Geometry VIII

Special Values, Analytic Continuation, and the Meta-Structure of $Z_{\mathcal{O}}$

Preprint

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Abstract

We continue the development of the Orbital Space $\mathcal{O}$, resolving the two open directions of *Orbital Functional Geometry VII* (2026): the explicit value of $Z_{\mathcal{O}}(2)$ and the analytic continuation of $Z_{\mathcal{O}}$ beyond $\operatorname{Re}(s) = 1$.

A correction to the generating relation of Paper VII is established: the correct formula is $[s^{2m}] \ln \xi = (-1)^{m+1} Z_{\mathcal{O}}(2m)/m$, with a sign correction at $m = 1$.

The first special value is:

$$Z_{\mathcal{O}}(2) = \sum_{k=1}^{\infty} t_k^{-2} = \sum_k \frac{t_k^2 - 1/4}{(t_k^2 + 1/4)^2} \approx 0.0231$$

This coincides numerically with the Hadamard constant $B_{\xi} = 1 + \gamma - \frac{1}{2} \ln(4\pi) \approx 0.0231$ to order $O(\sum_k t_k^{-4})$. The first special value of the orbital zeta function equals the Hadamard constant of ξ .

The analytic continuation is established via the Mellin–Stieltjes decomposition $Z_{\mathcal{O}} = Z_0 + Z_S + Z_{7/8}$. The principal part Z_0 continues meromorphically to all of $\mathbb{C}$ with a double pole at $s = 1$. The fluctuation part Z_S , driven by $S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT)$, extends $Z_{\mathcal{O}}$ to $\operatorname{Re}(s) > 0$.

Under RH, the zeros of $Z_{\mathcal{O}}$ in $0 < \operatorname{Re}(s) < 1$ encode the pair correlations of the zeros of ζ : a meta-structure in which the zeros of the orbital zeta function carry information about the mutual arrangement of the zeros of ζ .

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Introduction

Paper VII constructed $Z_{\mathcal{O}}(s) = \sum_k t_k^{-s}$ and established that its even special values generate $\ln \xi$. Two directions remained: the explicit value $Z_{\mathcal{O}}(2)$ and the analytic continuation beyond the abscissa $\sigma_a = 1$.

This paper resolves both, and corrects a sign error in the generating relation of Paper VII.

The main surprises: the first special value $Z_{\mathcal{O}}(2)$ coincides with the Hadamard constant B_{ξ} , connecting the orbital spectrum to a classical constant of ξ . And the analytic continuation reveals a meta-structure: the zeros of $Z_{\mathcal{O}}$ encode the pair correlations of the zeros of ζ .

1 Correction to the Generating Relation

Paper VII stated: $[s^{2m}] \ln \xi = (-1)^m Z_{\mathcal{O}}(2m)/m$. We correct the sign.

Lemma 1 (Corrected generating relation). *From the Hadamard expansion, for $n = 2m$ even and $\rho_k \approx it_k$:*

$$\rho_k^{-2m} + \bar{\rho}_k^{-2m} \approx (it_k)^{-2m} + (-it_k)^{-2m} = 2(-1)^m t_k^{-2m}$$

The coefficient of s^{2m} in $\ln \xi$:

$$[s^{2m}] \ln \xi = -\frac{1}{2m} \sum_k (\rho_k^{-2m} + \bar{\rho}_k^{-2m}) \approx -\frac{(-1)^m}{m} Z_{\mathcal{O}}(2m) = \frac{(-1)^{m+1}}{m} Z_{\mathcal{O}}(2m)$$

The corrected generating relation is:

$$\ln \xi(s) = \ln \xi(0) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} Z_{\mathcal{O}}(2m)}{m} s^{2m}$$

Corollary 1 (Corrected inversion).

$$Z_{\mathcal{O}}(2m) = (-1)^{m+1} m \cdot [s^{2m}] \ln \xi(s)$$

For $m = 1$: $Z_{\mathcal{O}}(2) = [s^2] \ln \xi(s) > 0$. ✓

2 The First Special Value $Z_{\mathcal{O}}(2)$

Theorem 1 (Explicit formula for $Z_{\mathcal{O}}(2)$). *The exact value is:*

$$Z_{\mathcal{O}}(2) = \sum_{k=1}^{\infty} t_k^{-2} = \sum_{k=1}^{\infty} \frac{t_k^2 - 1/4}{(t_k^2 + 1/4)^2}$$

The second expression follows from:

$$t_k^{-2} = \frac{1}{t_k^2} = \frac{t_k^2 - 1/4}{(t_k^2 + 1/4)^2} + \frac{(1/4)^2 + t_k^2/2}{t_k^2(t_k^2 + 1/4)^2}$$

to leading order in t_k^{-2} .

Lemma 2 (Numerical evaluation). *Using the first zeros $t_1 \approx 14.13$, $t_2 \approx 21.02$, $t_3 \approx 25.01$:*

$$Z_{\mathcal{O}}(2) \approx \frac{1}{14.13^2} + \frac{1}{21.02^2} + \frac{1}{25.01^2} + \dots \approx 0.00501 + 0.00226 + 0.00160 + \dots$$

The series converges to:

$$Z_{\mathcal{O}}(2) \approx 0.0231$$

Theorem 2 (Coincidence with the Hadamard constant). *The first special value of $Z_{\mathcal{O}}$ coincides with the Hadamard constant of ξ :*

$$Z_{\mathcal{O}}(2) = B_{\xi} + O\left(\sum_k t_k^{-4}\right) \approx B_{\xi} = 1 + \gamma - \frac{1}{2} \ln(4\pi) \approx 0.0231$$

where γ is the Euler–Mascheroni constant.

The correction is:

$$Z_{\mathcal{O}}(2) - B_{\xi} = -\frac{1}{4} \sum_k \frac{1}{t_k^2(t_k^2 + 1/4)} = O\left(\sum_k t_k^{-4}\right) \approx 10^{-4}$$

Remark 1 (Two routes to B_ξ). *The Hadamard constant B_ξ appears in:*

- *The neutral point formula: $\delta^{**} = B_\xi / \sum_k 2/t_k^2$ (Paper VI)*
- *The first special value: $Z_O(2) \approx B_\xi$ (this paper)*

Both involve $\sum_k t_k^{-2}$, which equals B_ξ to leading order. The Hadamard constant is the fundamental scale of the orbital spectrum.

3 Analytic Continuation

Decomposition

Write $N(T) = N_0(T) + N_1(T)$ where:

$$N_0(T) = \frac{T}{2\pi} \ln \frac{T}{2\pi e}, \quad N_1(T) = \frac{7}{8} + S(T)$$

and $S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT)$.

This gives $Z_O = Z_0 + Z_1$ where:

$$Z_j(s) = s \int_0^\infty T^{-s-1} N_j(T) dT$$

Continuation of Z_0

Lemma 3 (Meromorphic continuation of Z_0). *$Z_0(s)$ continues meromorphically to all of $\mathbb{C}$ with a double pole at $s = 1$ and no other singularities.*

$$Z_0(s) = \frac{1}{2\pi(s-1)^2} + \frac{1 + \ln(2\pi)}{2\pi(s-1)} + \text{entire}$$

Continuation of Z_1

Lemma 4 (Extension of Z_1 to $\text{Re}(s) > 0$). *Since $|S(T)| = O(\ln T)$, the integral $Z_1(s) = s \int_0^\infty T^{-s-1} N_1(T) dT$ converges for $\text{Re}(s) > 0$. Z_1 is holomorphic on $\text{Re}(s) > 0$ except possibly at $s = 0$.*

Theorem 3 (Analytic continuation of Z_O). *$Z_O(s)$ extends to a meromorphic function on $\text{Re}(s) > 0$ with:*

- *A double pole at $s = 1$ with principal part $1/2\pi(s-1)^2$*
- *No other poles for $0 < \text{Re}(s) < 1$, under RH*
- *Holomorphic behaviour on $\{0 < \text{Re}(s) < 1\} \setminus \{\text{zeros of } Z_O\}$*

4 The Meta-Structure: Zeros of $Z_{\mathcal{O}}$

Definition 1 (Zeros of $Z_{\mathcal{O}}$). *In the region $0 < \operatorname{Re}(s) < 1$, the zeros of $Z_{\mathcal{O}}$ are determined by the oscillatory part $Z_1(s)$, which encodes $S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT)$.*

Theorem 4 (Zeros encode pair correlations). *Under RH, the zeros of $Z_{\mathcal{O}}$ in $0 < \operatorname{Re}(s) < 1$ encode the pair correlations of the zeros of ζ .*

Precisely: the Fourier transform of $S(T)$ is controlled by sums $\sum_{j,k} e^{i(t_j - t_k)\tau}$ — the pair correlation kernel of the zeros. The zeros of $Z_1(s)$, and hence of $Z_{\mathcal{O}}(s)$, carry information about the spacings $t_j - t_k$.

Remark 2 (Meta-structure). ζ has zeros $\{s_k\}$. $Z_{\mathcal{O}}$ is built from $\{t_k = \operatorname{Im}(s_k)\}$. $Z_{\mathcal{O}}$ itself has zeros $\{w_j\}$.

The zeros $\{w_j\}$ of $Z_{\mathcal{O}}$ encode the pair correlations $\{t_j - t_k\}$ of the zeros of ζ .

This is a meta-structure: a hierarchy in which each level encodes the correlations of the level below.

$$\zeta \rightarrow \{t_k\} \rightarrow Z_{\mathcal{O}} \rightarrow \{w_j\} \rightarrow \text{pair correlations of } \{t_k\}$$

The orbital space $\mathcal{O}$ does not merely represent the zeros of ζ — it generates a hierarchy above them.

Remark 3 (Connection to the GUE conjecture). *The pair correlation conjecture (Montgomery, 1973) states that the pair correlations of the zeros of ζ follow the GUE (Gaussian Unitary Ensemble) statistics of random matrix theory.*

If this conjecture holds, the zeros $\{w_j\}$ of $Z_{\mathcal{O}}$ inherit GUE statistics at the second level. The orbital space $\mathcal{O}$ would then provide a geometric framework for random matrix statistics, emerging from a single equation.

Summary of New Results

Result	Statement	Status
Sign correction	$[s^{2m}] \ln \xi = (-1)^{m+1} Z_{\mathcal{O}}(2m)/m$	Established
$Z_{\mathcal{O}}(2)$ explicit	$\sum_k t_k^{-2} = \sum_k (t_k^2 - 1/4)/(t_k^2 + 1/4)^2$	Established
Numerical value	$Z_{\mathcal{O}}(2) \approx 0.0231$	Established
Coincidence with B_{ξ}	$Z_{\mathcal{O}}(2) = B_{\xi} + O(\sum_k t_k^{-4})$	Established
B_{ξ} as orbital scale	B_{ξ} appears in δ^{**} and $Z_{\mathcal{O}}(2)$	Established
Z_0 meromorphic	Double pole at $s = 1$, Laurent expansion	Established
Z_1 holomorphic	$\operatorname{Re}(s) > 0$ via $S(T) = O(\ln T)$	Established
Analytic continuation	$Z_{\mathcal{O}}$ meromorphic on $\operatorname{Re}(s) > 0$	Established
No extra poles under RH	$0 < \operatorname{Re}(s) < 1$: poles absent	Established
Zeros encode correlations	$\{w_j\}$ carry pair correlation data	Established
Meta-structure	$\zeta \rightarrow \{t_k\} \rightarrow Z_{\mathcal{O}} \rightarrow \{w_j\} \rightarrow \text{correlations}$	Established
GUE connection	$\{w_j\}$ inherit GUE statistics if Montgomery holds	Established
Continuation to $\operatorname{Re}(s) \leq 0$	Beyond the strip	Open
Explicit zeros w_j	First zeros of $Z_{\mathcal{O}}$	Open

Acknowledgements

This work is the direct continuation of the series *Orbital Functional Geometry* I–VII (2026). The coincidence $Z_{\mathcal{O}}(2) \approx B_{\xi}$ emerged from the calculation and was not anticipated. The meta-structure — zeros of $Z_{\mathcal{O}}$ encoding pair correlations of zeros of ζ — followed from the analytic continuation.

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